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ON DOMINANT CONTRACTIONS AND A GENERALIZATION OF THE ZERO-TWO LAW

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ABSTRACT. Zaharopol proved the following result: let $T, S : L^1(X, \mathcal{F}, \mu) \to L^1(X, \mathcal{F}, \mu)$ be two positive contractions such that $T \leq S$. If $\|S - T\| < 1$ then $\|S^n - T^n\| < 1$ for all $n \in \mathbb{N}$. In the present paper we generalize this result to multi-parameter contractions acting on L^1. As an application of that result we prove a generalization of the "zero-two" law.

Keywords: dominant contraction, positive operator, "zero-two" law.

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1. Introduction

Let $(X, \mathcal{F}, \mu)$ be a measure space with a positive σ-additive measure μ. In what follows for the sake of shortness by L^1 we denote the usual $L^1(X, \mathcal{F}, \mu)$ space associated with $(X, \mathcal{F}, \mu)$. A linear operator $T : L^1 \to L^1$ is called a *positive contraction* if $Tf \geq 0$ whenever $f \geq 0$ and $\|T\| \leq 1$.

In [9] it was proved so called "zero-two" law for positive contractions of L^1-spaces:

Theorem 1.1. *Let $T : L^1 \to L^1$ be a positive contraction. If for some $m \in \mathbb{N} \cup \{0\}$ one has $\|T^{m+1} - T^m\| < 2$, then*

$$\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0.$$

In [2] it was proved a "zero-two" law for Markov processes, which allowed to study random walks on locally compact groups. Other extensions and generalizations of the formulated law have been investigated by many authors [7, 4, 5].

Using certain properties of L^1-spaces Zaharopol [10] by means of the following theorem reproved Theorem 1.1.

Theorem 1.2. *Let $T, S : L^1 \to L^1$ be two positive contractions such that $T \leq S$. If $\|S - T\| < 1$ then $\|S^n - T^n\| < 1$ for all $n \in \mathbb{N}$*

In the paper we provide an example (see Example 2) for which the formulated theorem 1.2 can not be applied. Therefore, we prove a generalization of Theorem 1.2 for multi-parameter contractions acting on L^1. As a consequence

of that result we shall provide a generalization of the "zero-two" law. Similar generalization has been considered in [5].

2. Dominant operators

Let $T, S : L^1 \to L^1$ be two positive contractions. We write $T \le S$ if $S - T$ is a positive operator. In this case we have

$$\|Sx - Tx\| = \|Sx\| - \|Tx\|, \tag{2.1}$$

for every $x \ge 0$. Moreover, for positive operator $T : L^1 \to L^1$ one can prove the following equality

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1, x\ge 0} \|Tx\|. \tag{2.2}$$

The main result of this section is the following

Theorem 2.1. *Let $T_1, T_2, S_1, S_2 : L^1 \to L^1$ be positive contractions such that $T_i \le S_i$, $i = 1, 2$ and $S_1S_2 = S_2S_1$. If there is an $n_0 \in \mathbb{N}$ such that $\|S_1S_2^{n_0} - T_1T_2^{n_0}\| < 1$. Then $\|S_1S_2^n - T_1T_2^n\| < 1$ for every $n \ge n_0$.*

Proof. Let us assume that $\|S_1S_2^n - T_1T_2^n\| = 1$ for some $n > n_0$. Therefore, denote

$$m = \min\{n \in \mathbb{N} : \|S_1S_2^{n_0+n} - T_1T_2^{n_0+n}\| = 1\}.$$

It is clear that $m \ge 1$. The inequalities $T_1 \le S_1$, $T_2 \le S_2$ imply that $S_1S_2^{n_0+n} - T_1T_2^{n_0+n}$ is a positive operator. Then according to (2.2) there exists a sequence $\{x_n\} \in L^1$ such that $x_n \ge 0$, $\|x_n\| = 1, \forall n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \|(S_1S_2^{n_0+n} - T_1T_2^{n_0+n})x_n\| = 1. \tag{2.3}$$

Positivity of $S_1S_2^{n_0+n} - T_1T_2^{n_0+n}$ and $x_n \ge 0$ together with (2.1) imply that

$$\|(S_1S_2^{n_0+n} - T_1T_2^{n_0+n})x_n\| = \|S_1S_2^{n_0+m}x_n\| - \|T_1T_2^{n_0+m}x_n\| \tag{2.4}$$

for every $n \in \mathbb{N}$. It then follows from (2.3),(2.4) that

$$\lim_{n\to\infty} \|S_1S_2^{n_0+m}x_n\| = 1, \tag{2.5}$$

$$\lim_{n\to\infty} \|T_1T_2^{n_0+m}x_n\| = 0. \tag{2.6}$$

Thanks to the contractivity of S, Z and $S_1S_2 = S_2S_1$ one gets

$$\|S_1S_2^{n_0+m}x_n\| = \|S_2(S_1S_2^{n_0+m-1}x_n)\| \le \|S_1S_2^{n_0+m-1}x_n\| \le \|S_2^m x_n\|$$

which with (2.5) yields

$$\lim_{n\to\infty} \|S_1S_2^{n_0+m-1}x_n\| = 1, \quad \lim_{n\to\infty} \|S_2^m x_n\| = 1. \tag{2.7}$$

Moreover, the contractivity of S_i, T_i $(i = 1, 2)$ implies that $\|T_1T_2^{n_0+m-1}x_n\| \le 1$, $\|T_2^m x_n\| \le 1$ and $\|S_1S_2^{n_0}T^m x_n\| \le 1$ for every $n \in \mathbb{N}$. Therefore, we may

choose a subsequence $\{y_k\}$ of $\{x_n\}$ such that the sequences $\{\|T_1T_2^{n_0+m-1}y_k\|\}$, $\{\|T_2^m y_k\|\}$, $\{\|S_1S_2^{n_0}T^m x_k\|\}$ converge. Put

$$\alpha = \lim_{k\to\infty} \|T_1T_2^{n_0+m-1}y_k\|, \tag{2.8}$$

$$\beta = \lim_{k\to\infty} \|S_1S_2^{n_0}T^m y_k\|, \tag{2.9}$$

$$\gamma = \lim_{k\to\infty} \|T_2^m y_k\|. \tag{2.10}$$

The inequality $\|S_1S_2^{n_0+m-1} - T_1T_2^{n_0+m-1}\| < 1$ with (2.7) implies that $\alpha > 0$. Hence we may choose a subsequence $\{z_k\}$ of $\{y_k\}$ such that $\|T_1T_2^{n_0+m-1}z_k\| \neq 0$ for all $k \in \mathbb{N}$.

From $\|T_1T_2^{n_0+m-1}z_k\| \leq \|T_2^m z_k\|$ together with (2.8), (2.10) we find $\alpha \leq \gamma$, and hence $\gamma > 0$.

Using (2.1) one gets

$$\begin{aligned}
\|S_1S_2^{n_0}T_2^m z_k\| &= \|S_1S_2^{n_0+m}z_k - (S_1S_2^{n_0+m}z_k - S_1S_2^{n_0}T_2^m z_k)\| \\
&= \|S_1S_2^{n_0+m}z_k\| - \|S_1S_2^{n_0+m}z_k - S_1S_2^{n_0}T_2^m z_k\| \\
&\geq \|S_1S_2^{n_0+m}z_k\| - \|S_2^m z_k - T_2^m z_k\| \\
&= \|S_1S_2^{n_0+m}z_k\| - \|S_2^m z_k\| + \|T_2^m z_k\| \qquad (2.11)
\end{aligned}$$

Due to (2.5),(2.7) we have

$$\lim_{k\to\infty} \|S_1S_2^{n_0+m}z_k\| - \|S_2^m z_k\| = 0;$$

which with (2.11) implies that

$$\lim_{k\to\infty} \|S_1ZS_2^{n_0}T_2^m z_k\| \geq \lim_{k\to\infty} \|T_2^m z_k\|,$$

therefore, $\beta \geq \gamma$.

On the other hand, by $\|S_1S_2^{n_0}T_2^m z_k\| \leq \|T_2^m z_k\|$ one gets $\gamma \geq \beta$, hence $\gamma = \beta$.

Now set

$$u_k = \frac{T_2^m z_k}{\|T_2^m z_k\|}, \quad k \in \mathbb{N}.$$

Then using the equality $\gamma = \beta$ and (2.6) one has

$$\lim_{k\to\infty} \|S_1S_2^{n_0}u_k\| = \lim_{k\to\infty} \frac{\|S_1S_2^{n_0}T_2^m z_k\|}{\|T_2^m z_k\|} = 1,$$

$$\lim_{k\to\infty} \|T_1T_2^{n_0}u_k\| = \lim_{k\to\infty} \frac{\|T_1T^{n_0+m}z_k\|}{\|T_2^m z_k\|} = 0.$$

So, owing to (2.1) and positivity of $S_1S_2^{n_0} - T_1T_2^{n_0}$, we get

$$\lim_{k\to\infty} \|(S_1S_2^{n_0} - T_1T_2^{n_0})z_k\| = 1.$$

Since $\|u_k\| = 1, u_k \geq 0, \forall k \in \mathbb{N}$ from (2.2) one finds $\|S_1S_2^{n_0} - T_1T_2^{n_0}\| = 1$, which is a contradiction. This completes the proof. $\square$

Corollary 2.2. *Let $Z, T, S : L^1 \to L^1$ be positive contractions such that $T \leq S$ and $ZS = SZ$. If there is an $n_0 \in \mathbb{N}$ such that $\|Z(S^{n_0} - T^{n_0})\| < 1$. Then $\|Z(S^n - T^n)\| < 1$ for every $n \geq n_0$.*

Assume that $Z = Id$. If $n_0 = 1$, then from Corollary 2.2 we immediately get the Zaharopol's result (see Theorem 1.2). If $n_0 > 1$ then we obtain a main result of [8].

Let us provide an example of Z, S, T positive contractions for which statement of Corollary 2.2 is satisfied.

Example 1. Consider $\mathbb{R}^2$ with a norm $\|\mathbf{x}\| = |x_1| + |x_2|$, where $\mathbf{x} = (x_1, x_2)$. An order in $\mathbb{R}^2$ is defined as usual, namely $\mathbf{x} \geq 0$ if and only if $x_1 \geq 0$, $x_2 \geq 0$. Now define mappings $Z : \mathbb{R}^2 \to \mathbb{R}^2$,$T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$, respectively, by

$$Z(x_1, x_2) = (ux_1 + vx_2, ux_2), \tag{2.12}$$

$$S(x_1, x_2) = \left(\frac{x_1 + x_2}{2}, \frac{x_2}{2}\right), \tag{2.13}$$

$$T(x_1, x_2) = (\lambda x_2, 0). \tag{2.14}$$

The positivity of Z,S and T implies that $u, v, \lambda \geq 0$. It is easy to check that $T \leq S$ holds if and only if $2\lambda \leq 1$.

One can see that

$$\begin{aligned} \|Z\| = \sup_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x} \geq 0}} \|Z\mathbf{x}\| &= \max_{\substack{x_1+x_2=1 \\ x_1, x_2 \geq 0}} \{ux_1 + (u + v)x_2\} \\ &= \max_{0 \leq x_2 \leq 1} \{u + vx_2\} \\ &= u + v \end{aligned}$$

Hence, contractivity of Z implies that $u + v = 1$. Similarly, we find that $\|S\| = 1$ and $\|T\| = \lambda$. From (2.12) and (2.13) one gets that $ZS = SZ$.

By means of (2.12),(2.13),(2.14) one finds Similarly, one gets

$$\begin{aligned} \|Z(S - T)\| = \sup_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x} \geq 0}} \|Z(S - T)\mathbf{x}\| &= \max_{\substack{x_1+x_2=1 \\ x_1, x_2 \geq 0}} \left\{\frac{1}{2}\big(ux_1 + x_2 + ux_2 - 2\lambda ux_2\big)\right\} \\ &= \frac{1 + u(1 - 2\lambda)}{2}. \end{aligned} \tag{2.15}$$

The condition $2\lambda \leq 1$ yields that $\|Z(S - T)\| < 1$. Consequently, Corollary 2.2 implies $\|Z(S^n - T^n)\| < 1$ for all $n \in \mathbb{N}$.

Now let us formulate a multi-parametric version of Theorem 1.1.

Theorem 2.3. *Let $T_i, S_i : L^1 \to L^1$, $i = 1, \ldots, N$ be positive contractions such that $T_i \leq S_i$ with*

$$T_iT_j = T_jT_i, \quad S_iS_j = S_jS_i \quad \textit{for every } i, j = 1, \ldots, N. \tag{2.16}$$

If there are $n_{i,0} \in \mathbb{N}$, $i = 1, \ldots, N$ such that

$$\|S_1^{n_{1,0}} \cdots S_N^{n_{N,0}} - T_1^{n_{1,0}} \cdots T_N^{n_{N,0}}\| < 1. \tag{2.17}$$

Then

$$\|S_1^{m_1} \cdots S_N^{m_N} - T_1^{m_1} \cdots T_N^{m_N}\| < 1 \tag{2.18}$$

for all $m_i \geq n_{i,0}$, $i = 1, \ldots, N$.

Proof. Let us fix the first $N-1$ operators in (2.17), i.e. for a moment we denote

$$\mathbf{S}_{N-1} = S_1^{n_{1,0}} \cdots S_{N-1}^{n_{N-1,0}} \quad \mathbf{T}_{N-1} = T_1^{n_{1,0}} \cdots T_{N-1}^{n_{N-1,0}}, \tag{2.19}$$

then (2.17) can be written as follows

$$\|\mathbf{S}_{N-1} S_N^{n_{N,0}} - \mathbf{T}_{N-1} T_N^{n_{N,0}}\| < 1.$$

After applying Theorem 2.1 to the last inequality we find

$$\|\mathbf{S}_{N-1} S_N^{m_N} - \mathbf{T}_{N-1} T_N^{m_N}\| < 1 \tag{2.20}$$

for all $m_N \geq n_{N,0}$. Now taking into account (2.19) and (2.16) we rewrite (2.20) as follows

$$\|S_N^{m_N} S_1^{n_{1,0}} \cdots S_{N-1}^{n_{N-1,0}} - T_N^{m_N} T_1^{n_{1,0}} \cdots T_{N-1}^{n_{N-1,0}}\| < 1. \tag{2.21}$$

Now again applying the same idea as above to (2.21) we get

$$\|S_N^{m_N} S_1^{n_{1,0}} \cdots S_{N-1}^{m_{N-1}} - T_N^{m_N} T_1^{n_{1,0}} \cdots T_{N-1}^{m_{N-1}}\| < 1,$$

for all $m_{N-1} \geq n_{N-1,0}, m_N \geq n_{N,0}$. Hence, continuing this procedure $N-2$ times we obtain the desired inequality. □

Remark 3.1. It should be noted the following:

(i) Since the dual of L^1 is L^∞ then due to the duality theory the proved Theorems 2.1 and 2.3 holds true if we replace L^1-space with L^∞.

(ii) Unfortunately, that the proved theorems and its corollaries are not longer true if one replaces L^1-space by an L^p-space, $1 < p < \infty$. Indeed, consider $X = \{1, 2\}$, $\mathcal{F} = \mathcal{P}(\{1, 2\})$ and the measure μ is given by $\mu(\{1\}) = \mu(\{2\}) = 1/2$. In this case, L^p is isomorphic to the Banach lattice $\mathbb{R}^2$ (here an order is defined as usual, namely $\mathbf{x} \geq 0$ if and only if $x_1 \geq 0$, $x_2 \geq 0$) with the norm $\|\mathbf{x}\|_p = \big(|x_1|^p + |x_2|^p\big)^{1/p}/2$, where $\mathbf{x} = (x_1, x_2)$. Define two operators by

$$S(x_1, x_2) = \left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}\right), \quad T(x_1, x_2) = \left(0, \frac{x_1}{2}\right)$$

Then it is shown (see [10]) that $\|S - T\| < 1$, but $\|S^2 - T^2\| = 1$.

(iii) It would be better to note that certain ergodic properties of dominant positive operators has been studied in [3]. In general, a monograph [6] is devoted to dominant operators.

Let us give another example, for which conditions of Theorem 1.2 does not hold, but Theorem 2.1 can be applied.

Example 2. Let us consider $\mathbb{R}^2$ as in Example 1. Now define mappings $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ as follows

$$S(x_1, x_2) = \left(\frac{1}{2}x_1 + \frac{1}{3}x_2, \frac{1}{2}x_1 + \frac{1}{3}x_2\right), \tag{2.22}$$

$$T(x_1, x_2) = \left(\frac{1}{4}x_2, 0\right). \tag{2.23}$$

It is clear that S and T are positive and $T \leq S$.

One can see that $\|S\| = 1$, $\|T\| = 1/4$. From (2.22),(2.23) one gets

$$\|S - T\| = \sup_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\geq 0}} \|(S - T)\mathbf{x}\| = \max_{0\leq x_1\leq 1} \left\{\frac{7x_1 + 5}{12}\right\} = 1 \tag{2.24}$$

$$\|S^2 - T^2\| = \sup_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\geq 0}} \|(S^2 - T^2)\mathbf{x}\| = \max_{0\leq x_1\leq 1} \left\{\frac{5x_1 + 10}{18}\right\} = \frac{15}{18} \tag{2.25}$$

Consequently, we have positive contractions T and S with $S \geq T$ such that $\|S - T\| = 1$,$\|S^2 - T^2\| < 1$. This shows that the condition of Theorem 1.2 is not satisfied, but due to Corollary 2.2 with $Z = id$ we have $\|S^n - T^n\| < 1$ for all $n \geq 2$. Therefore the proved Theorem 2.2 is an extension of the Zaharopol's result.

3. A generalization of the zero-two law

In this section we are going to prove a generalization of the zero-two law for positive contractions on L^1. Before formulate the main result we prove some auxiliary facts.

First note that for any $x, y \in L^1$ one defines

$$x \wedge y = \frac{1}{2}(x + y - |x - y|). \tag{3.1}$$

It is well known (see [1]) that for any mapping S of L^1 one can define its modulus by

$$|S|x = \sup\{Sy : \ |y| \leq x\}, \quad x \in L^1, x \geq 0. \tag{3.2}$$

Hence, similarly to (3.1) for given two mappings S, T of L^1 we define

$$(S \wedge T)x = \frac{1}{2}(Sx + Tx - |S - T|x), \quad x \in L^1. \tag{3.3}$$

A linear operator $Z : L^1 \to L^1$ is called *a lattice homomorphism* whenever

$$Z(x \vee y) = Zx \vee Zy \tag{3.4}$$

holds for all $x, y \in L^1$. One can see that such an operator is positive. Note that such homomorphisms were studied in [1].

Recall that a net $\{x_\alpha\}$ in L^1 is *order convergent* to x, denoted $x_\alpha \to^o x$ whenever there exists another net $\{y_\alpha\}$ with the same index set satisfying $|x_\alpha - x| \leq y_\alpha \downarrow 0$. An operator $T : L^1 \to L^1$ is said to be *order continuous*, if $x_\alpha \to^o 0$ implies $Tx_\alpha \to^o 0$.

Lemma 3.1. *Let S, T be positive contractions of L^1, and Z be an order continuous lattice homomorphism of L^1. Then one has*

$$Z|S - T| = |Z(S - T)|. \tag{3.5}$$

Moreover, we have

$$Z(S \wedge T) = ZS \wedge ZT. \tag{3.6}$$

Proof. From (3.2) we find that

$$\begin{aligned} Z|S - T|x &= Z(\sup\{(S - T)y : \ |y| \leq x\}) \\ &= \sup\{Z(S - T)y : \ |y| \leq x\}) \\ &= |Z(S - T)|x, \end{aligned} \tag{3.7}$$

for every $x \in L^1, x \geq 0$.

The equality (3.3) yields that

$$\begin{aligned} Z|S - T| = ZS + ZT - 2Z(S \wedge T), \\ |Z(S - T)| = ZS + ZT - 2(ZS \wedge ZT), \end{aligned} \tag{3.8}$$

which with (3.7) imply that

$$Z(S \wedge T) = ZS \wedge ZT.$$

□

In what follows, an order continuous lattice homomorphism $Z : L^1 \to L^1$ with $\|Z\| \leq 1$, is called *a lattice contraction*.

Now we have the following

Lemma 3.2. *Let Z be a lattice contraction and T be a positive contraction of L^1 such that $ZT = TZ$. If for some $m \in \mathbb{N} \cup \{0\}$, $k \in \mathbb{N}$ one has $\|Z(T^{m+k} - T^m)\| < 2$, then $\|Z(T^{m+k} - T^{m+k} \wedge T^m)\| < 1$.*

Proof. According to the assumption there is $\delta > 0$ such that $\|Z(T^{m+k} - T^m)\| = 2(1-\delta)$. Let us suppose that $\|Z(T^{m+k} - T^{m+k} \wedge T^m)\| = 1$. Then thanks to (2.2) there exists $x \in L^1$ with $x \geq 0$, $\|x\| = 1$ such that

$$\|Z(T^{m+k} - T^{m+k} \wedge T^m)x\| > 1 - \frac{\delta}{4},$$

which with (2.1) implies that $\|ZT^{m+k}x\| > 1 - \delta/4$ and $\|Z(T^{m+k} \wedge T^m)x\| < \delta/4$. The commutativity T and Z yields that $\|ZT^m x\| > 1 - \delta/4$.

Now using (3.8) and (3.6) one finds

$$\begin{aligned}
\big\||Z(T^{m+k} - T^m)|x\big\| &= \|ZT^{m+k}x\| + \|ZT^m x\| - 2\|Z(T^{m+k} \wedge T^m)x\| \\
&> 1 - \frac{\delta}{4} + 1 - \frac{\delta}{4} - 2 \cdot \frac{\delta}{4} \\
&= 2\left(1 - \frac{\delta}{2}\right).
\end{aligned}$$

This with the equality

$$\big\||Z(T^{m+k} - T^m)|\big\| = \|Z(T^{m+k} - T^m)\|,$$

contradicts to $\|Z(T^{m+k} - T^m)\| = 2(1 - \delta/2)$. $\square$

Lemma 3.3. *Let Z be a lattice contraction and T be a positive contraction of L^1 such that $ZT = TZ$. If for some $m \in \mathbb{N} \cup \{0\}$, $k \in \mathbb{N}$ one has $\|Z(T^{m+k} - T^{m+k} \wedge T^m)\| < 1$, then for any $\varepsilon > 0$ there are $d, n_0 \in \mathbb{N}$ such that*

$$\|Z^d(T^{n+k} - T^n)\| < \varepsilon \quad \textit{for all } n \geq n_0$$

Proof. It is known that (see [11], p. 310) for any contraction T on L^1 there is $\gamma > 0$ such that

$$\left\|\left(\frac{I+T}{2}\right)^\ell - T\left(\frac{I+T}{2}\right)^\ell\right\| \leq \frac{\gamma}{\sqrt{\ell}}. \tag{3.9}$$

Then for given $k \in \mathbb{N}$, using (3.9) one easily finds that

$$\left\|\left(\frac{I+T}{2}\right)^\ell - T^k\left(\frac{I+T}{2}\right)^\ell\right\| \leq \frac{k\gamma}{\sqrt{\ell}}. \tag{3.10}$$

Let $\varepsilon > 0$ and fix $\ell \in \mathbb{N}$ such that $k\gamma/\sqrt{\ell} < \varepsilon/4$.

Then according to Corollary 2.2 from the assumption of the lemma we have

$$\big\|Z(T^{\ell(m+k)} - (T^{m+k} \wedge T^m)^\ell)\big\| < 1. \tag{3.11}$$

Hence,

$$
\begin{aligned}
\left\|Z\left(T^{\ell(m+k)}\right.\right. &- \left.\left.\left(\frac{I+T}{2}\right)^{\ell}(T^{m+k}\wedge T^m)^{\ell}\right)\right\| = \\
&= \left\|Z\left(T^{\ell(m+k)} - \frac{1}{2^\ell}\sum_{i=0}^{\ell} C_\ell^i T^i (T^{m+k}\wedge T^m)^{\ell}\right)\right\| \\
&\leq \sum_{i=0}^{\ell} \frac{C_\ell^i}{2^\ell}\|Z(T^{\ell(m+k)} - T^i(T^{m+k}\wedge T^m)^{\ell})\| \\
&\leq \frac{1}{2^\ell}\|Z(T^{\ell(m+k)} - (T^{m+k}\wedge T^m)^{\ell})\| + \sum_{i=0}^{\ell}\frac{C_\ell^i}{2^\ell} \\
&< \frac{1}{2^\ell} + \sum_{i=1}^{\ell}\frac{C_\ell^i}{2^\ell} = 1. \qquad (3.12)
\end{aligned}
$$

Define

$$Q_\ell := T^{\ell(m+k)} - \left(\frac{I+T}{2}\right)^{\ell}(T^{m+k}\wedge T^m)^{\ell}$$

and put $V_\ell^{(1)} = (T^{m+k}\wedge T^m)^{\ell}$. Then one can see that

$$T^{\ell(m+k)} = \left(\frac{I+T}{2}\right)^{\ell} V_\ell^{(1)} + Q_\ell.$$

Now for every $d\in\mathbb{N}$, define

$$V_\ell^{(d+1)} = T^{\ell(m+k)}V_\ell^{(d)} + V_\ell^{(1)}Q_\ell^d.$$

Then by induction one can establish [11] that

$$T^{d\ell(m+k)} = \left(\frac{I+T}{2}\right)^{\ell} V_\ell^{(d)} + Q_\ell^d \qquad (3.13)$$

for every $d\in\mathbb{N}$.

Due to Proposition 2.1 [10] one has

$$\|V_\ell^{(d)}\| \leq 2 \qquad (3.14)$$

for all $d\in\mathbb{N}$.

Now from (3.12) we find $\|ZQ_\ell\| < 1$, therefore there exists $d\in\mathbb{N}$ such that $\|(ZQ_\ell)^d\| < \varepsilon/4$. So, commutativity Z and T implies that $ZQ_\ell = Q_\ell Z$, which yields that $\|Z^dQ_\ell^d\| < \varepsilon/4$.

Put $n_0 = d\ell(m+k)$, then from (3.13) with (3.10),(3.14) we get

$$\begin{aligned}
\|Z^d(T^{n_0+k} - T^{n_0})\| &= \left\| Z^d\left(T^k\left(\frac{I+T}{2}\right)^\ell - \left(\frac{I+T}{2}\right)^\ell\right)V_\ell^{(d)} \right. \\
&\quad \left. + Z^d(T^k Q_\ell^d - Q_\ell^d)\right\| \\
&\le \left\|\left(T^k\left(\frac{I+T}{2}\right)^\ell - \left(\frac{I+T}{2}\right)^\ell\right)V_\ell^{(d)}\right\| \\
&\quad + \|Z^d Q_\ell^d (T-1)\| \\
&\le 2\cdot\frac{k\gamma}{\sqrt{\ell}} + 2\cdot\frac{\varepsilon}{4} < \varepsilon.
\end{aligned}$$

Take any $n \ge n_0$, then from the last inequality one finds

$$\|Z^d(T^{n+k} - T^n)\| = \|T^{n-n_0}Z^d(T^{n_0+k} - T^{n_0})\| \le \|Z^d(T^{n_0+k} - T^{n_0})\| < \varepsilon$$

which completes the proof. □

Now we are ready to formulate the main result of this section.

Theorem 3.4. *Let Z, T be two positive contractions of L^1 such that $TZ = ZT$. If for some $m \in \mathbb{N}\cup\{0\}$, $k \in \mathbb{N}$ one has $\|Z(T^{m+k} - T^m)\| < 2$, then for any $\varepsilon > 0$ there are $d, n_0 \in \mathbb{N}$ such that*

$$\|Z^d(T^{n+k} - T^n)\| < \varepsilon \quad \textit{for all } n \ge n_0$$

The proof of this theorem immediately follows from Lemmas 3.2 and 3.3.

Remark. Note that if we take as $Z = I$,$k = 1$ then we obtain Theorem 1.1 as a corollary of Theorem 3.4.

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